Probability Generating Function

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1 Random Variable with Finite Support

Let X be a random variable that takes possible values $0, 1, \cdots, N$. Define G_X be the PGF of X by putting the probabilities as the coefficients of a polynomial:

$$G_X(s) = p_0 + p_1 s + \cdots + p_N s^N = \sum_{i=0}^{N} p_i s_i$$

Let see why PGF might be useful:

1.1 Probability: Evaluating at 0

Set s to be zero. That is,

$$G_X(0) = p_0$$

If we take the first derivative of G, we get

$$G_X^{(1)}(s) = p_1 + 2p_2 s + \cdots + Np_N s^{N-1}$$

Then,

$$G_X^{(1)}(0) = p_1$$

Generalize the observation, we get

$$G^{(i)}(0) = p_i$$

1.2 Expectation: Evaluating at 1

Let's take another look at $G_X^{(1)}(s)$, which expands out to be $p_1 + 2p_2 s + \cdots + Np_N s^{N-1}$. Recall the expectation of X is the weighted average:

$$E[X] = \sum xP(X = x) = \sum ip_i = p_1 + 2p_2 + \cdots + Np_N s^{N-1}$$

Notice that the expectation is exactly $G_X^{(1)}(s)$ evaluated at $s = 1$. In other words,

$$E[X] = G_X^1(1)$$

Now, consider $G_X^{(2)}(s)$.

$$\begin{aligned} G_X^{(2)}(s) &= (G_X^{(1)}(s))' \\ &= (p_1 + 2p_2 s + \cdots + Np_N s^{N-1})' \\ &= (2\cdot 1)p_2 + (3\cdot 2)p_3 s + \cdots + (N\cdot(N-1))p_N s^{N-2} \end{aligned}$$

Evaluating at $s = 1$, we get

$$G_X^{(2)}(1) = (2\cdot 1)p_2 + (3\cdot 2)p_3 + \cdots + (N(N-1))p_N \quad (1)$$
$$= E[X(X-1)] \quad (2)$$

1.3 Variance

From the previous section, we learned that $G_X^{(2)}(1) = E[X(X-1)]$. Thus, we know

$$G_X^{(2)}(1) = E[X(X-1)] = E[X^2 - X)] = E[X^2] - E[X]$$

This reminds us the formula for $\mathrm{Var}(X)$

$$Var(X) = E[X^2] - (E[X])^2$$

Luckily, we know $E[X]$ in terms of G_X from the previous section. Then,

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= (G_X^{(2)}(1) + E[X]) - (E[X])^2 \\ &= G_X^{(2)}(1) + G_X^{(1)}(1) - (G_X^{(1)}(1))^2 \end{aligned}$$

2 Random Variable with Infinite Support

Let's extend our definition of PGF to random variables with infinitely many non-negative integer values as follows

$$G_X(s) = p_0 + p_1 s + p_2 s^2 + \cdots = \sum_{i=0}^{\infty} p_i s^i$$

, where $-1 < s < 1$.

The following sections further explore some interesting properties of PGF using a Geometric random variable as an example.

2.1 PGF of Geometric Random Variable

You might want to ask, why do we ever want to use PGF? If we already know the distribution of X, why do we bother constructing a polynomial based on probabilities?

Those are very reasonable questions. However, the PGF of some randm variables, for example geometric random variable, have nice form so that it would be easier to get the expectation or variance by taking the derivatives.

Consider $X \sim \text{Geometric}(p)$. The PGF of X has a nice form,

$$\begin{aligned}G_X(s) &= \sum_{i=0}^{\infty} p_i s^i \\ &= p_0 + p_1 s + p_2 s^2 + \cdots \\ &= p + (1-p)ps + (1-p)^2 ps^2 + \cdots \\ &= p + p(qs) + p(q^2 s^2) + \cdots \\ &= p \sum_{i=0}^{\infty} (qs)^i\end{aligned}$$

, where $q = 1 - p$. Since $q, s < 1$, we can apply the formula for sum of infinite geometric series.

$$G_X(s) = \frac{p}{1-qs}$$

Beautiful!

2.2 Expectation and Variance of Geometric Random Variable

The formulae that we derived in section 1 still holds for random variables with infinity support. Then, expectation and variance of a geometric random variable can be calculated by taking the derivative of G_X.

$$G_X^{(1)}(s) = (\frac{p}{1-qs})' = \frac{pq}{(1-qs)^2}$$

$$G_X^{(1)}(1) = \frac{pq}{(1-q)^2} = \frac{p(1-p)}{p^2} = \frac{1-p}{p}$$

$$G_X^{(2)}(s) = (\frac{pq}{(1-qs)^2})' = \frac{2pq^2}{(1-qs)^3}$$

$$G_X^{(2)}(1) = \frac{2pq^2}{(1-q)^3} = \frac{2pq^2}{p^3} = \frac{2(1-p)^2}{p^2}$$

Thus,

$$E[X] = G_X^{(1)}(1) = \frac{1-p}{p}$$

$$\begin{aligned}Var(X) &= G_X^{(2)}(1) + G_X^{(1)}(1) - (G_X^{(1)}(1))^2 \\ &= \frac{2(1-p)^2}{p^2} + \frac{1-p}{p} - (\frac{1-p}{p})^2 \\ &= \frac{1-p}{p^2}\end{aligned}$$